Utilizing the Hermite-Ostrogradski Formula for the Integration of Rational Functions

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ABSTRACT

The purpose of this report is to provide a clear and detailed presentation of the formula for the integration of rational functions known as the Hermite-Ostrogradski method. The proof of the formula will be given and applied to examples, demonstrating how this method of integration can prove superior to traditional method in certain situations. Also, the implementation of the formula utilizing computer algebra systems will be discussed as well as an additional method for hand computation The significance of this lies in the fact that it will unearth a largely forgotten result and will provide the reader with a more efficient method to integrate a large class of functions. A theoretical result and its proof is also included that permits one to recognize a large class of rational functions that can not be integrated in closed form.

KEYWORDS: Integration, Rational Functions

1. INTRODUCTION

The integration of rational functions (quotients of polynomials) is a frequently occurring task in Mathematics and Physics. Many integrals can be reduced to integrals of rational functions. All standard Calculus texts devote a substantial amount of space to this topic, underlying its importance, and at the same time exhibiting the extensive computation needed for the task. Most examples require partial fraction decomposition and integration of each partial fraction. With these involved computations also come increasing opportunities for mistakes, which can prove daunting.

Although the Hermite-Ostrogradski formula for the integration of rational numbers has been around for some time, it does not appear to be well known nor does it appear in Calculus books. This method of integration is beneficial in that it enables the user to find the rational part of an integral without any integration, leaving only the transcendental portion to be integrated, and the factorization of the denominator does not have to be known. Also, much of the calculation can be performed using a computer algebra system or even some graphing calculators.

In this report, I will provide the proof of the Hermite-Ostrogradski formula as presented in the October 1992 edition of *The American Mathematical Monthly* by T.N. Subramaniam and Donald E. G. Malm in the article, “How to Integrate Rational Functions” which is the main reference. I will also discuss application of the formula with and without computer algebra systems, showing examples.

2. THE PROOF [2]

Let P/Q be a rational function. Let $Q = \prod_{i=1}^{n} h_i^{\alpha_i}$ be the factorization of Q into linear and irreducible quadratic factors, and let $Q_1 = \prod_{i=1}^{n} h_i^{\alpha_i - 1}$ and $Q_2 = \prod_{i=1}^{n} h_i$. Then there exist polynomials P_1 and P_2 such that

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx \qquad (1)$$

In order to prove this, two cases must be considered. The first is when Q has one distinct repeated irreducible factor that takes the form $Q(x) = (x - c)^m$ or $Q(x) = (x^2 + ax + b)^m$ with the quadratic being irreducible and $m \geq 1$.

When $Q(x) = (x - c)^m$, the integral becomes $\int \frac{P(x)}{(x-c)^m}\,dx$, where $P(x)$ is a polynomial. When $P(x)$ is written as $P(x) = \sum_{k=0}^{n} a_k (x-c)^k$, then

$$\int \frac{P(x)}{(x-c)^m}\,dx = \sum_{k=-m}^{n-m} a_{k+m} \int (x-c)^k\,dx.$$

Because $Q_2 = (x - c)$ and the integrated terms have the common denominator $Q_1 = (x - c)^{m-1}$, we get to equation (1) by integrating all terms except for when $k = -1$.

When $Q(x) = (x^2 + ax + b)^m$, $P(x)$ is divided by $Q_2(x) = (x^2 + ax + b)$. Thus $P(x) = R(x)Q_2(x) + S(x)$, where $S(x)$ is linear. Then,

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \int \frac{S(x)}{Q_2(x)^m}\,dx$$

Using a standard reduction formula that can be obtained through integration by parts, we get to

$$\int \frac{Ax + B}{(x^2 + ax + b)^m}\,dx = \frac{M(x)}{(x^2 + ax + b)^{m-1}} + \int \frac{N}{(x^2 + ax + b)^{m-1}}\,dx,$$

where $M(x)$ is a linear polynomial and N is a constant. Through repeated application of this formula, we achieve

$$\int \frac{Ax + B}{(x^2 + ax + b)^m}\,dx = \frac{M(x)}{(x^2 + ax + b)^{m-1}} + \int \frac{N}{x^2 + ax + b}\,dx,$$

where $M(x)$ is a polynomial and N is a constant. From here we can get

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \frac{M(x)}{Q_2(x)^{m-1}} + \int \frac{N}{Q_2(x)}\,dx.$$

The same process applied to $\int R(x) / Q_2(x)^{m-1} dx$ will lead to the achievement of equation (1).

The second case to consider is when Q has two or more distinct irreducible factors. Assume equation (1) holds true for $k < K$, $K > 1$. Let $Q(x)$ have K distinct irreducible factors and $Q(x) = \prod_{i=1}^{K} h_i^{\alpha_i}$ be the irreducible factorization of Q. Because h_1 and $\prod_{i=2}^{K} h_i(x)^{\alpha_i} = g(x)$ are relatively prime, there exist polynomials $a(x)$ and $b(x)$ where $P(x) = a(x)h_1(x)^{\alpha_1} + b(x)g(x)$, and

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{a(x)}{g(x)}\,dx + \int \frac{b(x)}{h_1(x)^{\alpha_1}}\,dx.$$

By using induction on the first case, each of the integrals on the right side of the equation can be written in the form of equation (1).

In the case that degree $P <$ degree Q, P_1 and P_2 can be found so that degree $P_1 <$ degree Q_1 and degree $P_2 <$ degree Q_2. If degree $P_2 \geq$ degree Q_2, divide P_2 by Q_2, integrate the quotient and absorb it into P_1 / Q_1. If degree $P_1 =$ degree Q_1, P_1 / Q_1 is a constant added to a proper rational function. The constant can then be dropped from the equation. Last, if degree $P_1 >$ degree Q_1, P_1 / Q_1 is a polynomial of degree ≥ 1 and a proper rational function. Then the limit at infinity on the right side of equation (1) would not be zero, which cannot occur. It is important to note that the final integral in equation (1) is a sum of logarithms and arctangents.

3. APPLICATION

Having given the proof of the formula, it is important to explain the real benefit of the Hermite-Ostrogradski formula [2]. $Q_1$and Q_2 can be found without factoring Q. Q_1 = g.c.d.(Q, Q') and Q_2 = Q/Q_1. Then it follows that Q_1 divides $Q_1'Q_2$ and $S = Q_1'Q_2/ Q_1$ is a polynomial. By differentiating both sides of equation (1),

$$P/Q = \frac{Q_1P_1' - P_1Q_1'}{Q_1^2} + \frac{P_2}{Q_2}$$

$$= \frac{P_1' - P_1Q_1'/Q_1}{Q_1} + \frac{P_2}{Q_2}$$

After multiplying through by Q, we get $P= P_1'Q_2 - P_1S + P_2Q_1$. Since P_1, Q_1, Q_2 and S are known, P_1 and P_2 can be solved for by the method of undetermined coefficients.

Although this is equivalent to the amount of calculation in the partial fractions method, there is no integration left to be completed for the rational part of the integral.

Here is where the computer algebra systems could be utilized.

Input: Polynomials P and Q, with degree P< degree Q.

Output: P_1 / Q_1 and P_2 / Q_2.

(1) Q_1:=g.c.d.(Q, Q'); Q_2:=Q/Q_1
(2) S:= $Q_1'Q_2/ Q_1$
(3) q:= degree Q_1; p := degree Q_2
(4) Write $P_1(x) := A_{q-1}x^{q-1} + A_{q-2}x^{q-2} + \cdots + A_0$ and $Q_1(x) := B_{p-1}x^{p-1} + B_{p-2}x^{p-2} + \cdots + B_0$
(5) Compute T:= $P_1'Q_2 - P_1S + P_2Q_1$
(6) Equate the coefficients of T to those of P
(7) Solve the linear system of equations for the unknowns A_i and B_i.

If d = degree Q, then in step 7 there will be d equations, resulting in d unknowns.

If the computation is done by hand, the g.c.d. of Q and Q_1can be found by using a zero-row reduction formula [1]. First, let $d(x)$ = g.c.d.(Q , Q_1), $Q = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $Q_1 = b_0 + b_1x + b_2x^2 + \ldots + b_tx^t$ with real coefficients and $a_n \neq 0$, $b_t \neq 0$, $t \leq n$. Since $d(x)$ = g.c.d.(Q , Q_1), the following properties are true. 1.) $d(x)$ is the g.c.d. of $k_1Q(x)$ and $k_2Q_1(x)$ for nonzero real k_1, k_2. 2.) $d(x)$ is the g.c.d. of $Q(x) + kQ_1(x)$ and $Q_1(x)$ for real k. 3.) If $a_0 = 0$ and $b_0 \neq 0$, then $d(x)$) is the g.c.d. of $Q(x)$ and $Q_1(x)/x$. Allow $Q(x)$ and $Q_1(x)$ to be represented by the 2 by $n + 1$ matrix C:

$$C = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{bmatrix}.$$

Initially, any leading columns of zeros are removed from the matrix C to obtain a matrix C' with the first column nonzero. If the first K columns are zero, by property 4, x^K is a factor of $d(x)$.and the remainder of the g.c.d. can be computed using C'. Normalization of a row is to be defined as the operation that shifts a row to the left as many positions as necessary to remove all leading zeros and divide the row by its leading coefficient. By properties 1 and 3, each row of C' can be normalized, and the g.c.d of the polynomials represented by the new rows will be the same as $d(x)/ x^K$. By property 2, the first entry in row 2 can be changed to zero by replacing row 2 with row 2 minus row 1. Then by repeating the normalization process and replacement process, until one of the rows consists entirely of zeros, $d(x)/ x^K$ is represented by the remaining nonzero row of the matrix.

4. EXAMPLES

Using the Hermite-Ostrogradski formula,

1. The first example comes from Subramaniam [2].

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1}\,dx$$

$$Q_1 = g.c.d.$$

$$(x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1,$$
$$6x^5 + 10x^4 + 12x^3 + 12x^2 + 6x + 2)$$
$$= x^3 + x^2 + x + 1$$
$$Q_2 = Q/Q_1 = x^3 + x^2 + x + 1$$
$$P_1 = Ax^2 + Bx + C, \qquad P_2 = Dx^2 + Ex + F$$
$$T = P_1'Q_2 - P_1S + P_2Q_1$$

$$= Dx^5 + (-A + D + E)x^4$$
$$+ (-2B + D + E + F)x^3$$
$$+(A - B - 3C + D + E + F)x^2$$
$$+ (2A - 2C + E + F)x$$
$$+(B - C + F).$$

By equating the coefficients of the resulting T with the coefficients of P and solving for A, B, C, D, E, and F, the result becomes

$$-\frac{x^2 - x + 4}{x^3 + x^2 + x + 1} + \int \frac{3x + 3}{x^3 + x^2 + x + 1}\, dx.$$

Then factor $x + 1$ out of the remaining integral making the last term

$$3\int \frac{dx}{x^2 + 1} = 3\tan^{-1} x.$$

2. The second example also demonstrates the usefulness of the Hermite-Ostrogradski formula. This example cannot be solved by traditional methods, but through the use of the formula,

$$\int \frac{2x^8 + 8x^5 + 15x^4 - 24x^3 +}{(2x^5 - 15x + 6)^2}\, dx$$

$Q_1 = Q_2 = 2x^5 - 15x + 6$, and

$P_1 = -x^4 - x + 3$, $P_2 = 0$

The result becomes

$$\frac{-x^4 - x + 3}{2x^5 - 15x + 6}$$

3. The third example shows denominator degree > 2 and $Q_1 \neq Q_2$.

$$\int \frac{-x^4 - 4x^3 + 3x^2 - 16x}{(x^2 + 1)^3}\, dx$$

$Q_1 = x^4 + 2x^2 + 1$, $Q_2 = x^2 + 1$, and
$P_1 = x^3 + 2x^2 + 5$, $P_2 = 0$
The result becomes

$$\int \frac{x^3 + 2x^2 + 5}{(x^2 + 1)^2}\, dx$$

It is important to note here that most integrals of rational functions lead to extensive computation when using this method by hand. Through attempting both hand computation and utilizing a computer algebra system, it becomes apparent why the ability to utilize computers to apply the Hermite-Ostrogradski formula is very appealing.

5. CLOSED FORMULA FOR THE TRANSCENDETAL PART [2]

If the roots of the denominator are known, then the transcendental part of the integral can be expressed as

$$\int P(x)/Q(x)\, dx = \sum \frac{P(a)}{Q'(a)} \operatorname{Log}(x - a) \qquad (2)$$

with the sum ranging over all the roots a of $Q(x)$, including complex roots. In deriving this formula, it is assumed that Q does not have repeated roots and that degree P < degree Q. Let a be a root of $Q(x)$, and let $Q(x) = (x - a)Q_1(x)$, with $Q_1(a) \neq 0$. $Q_1(x)$ is now used with a different meaning. Let A be a constant and $A = P(a) / Q_1(a)$. Then

$$P/Q = \frac{A}{x - a} + \frac{P_1(x)}{Q_1(x)}$$

and

$$P_1(x) = Q_1(x)\left\{\frac{P(x)}{Q(x)} - \frac{P(a)}{Q_1(a)}\frac{1}{x - a}\right\}$$
$$= \frac{1}{x - a}\left\{P(x) - \frac{P(a)}{Q_1(a)}Q_1(x)\right\}$$

$P_1(x)$ is also used with a different meaning. Since $P(x) - (P(a) / Q_1(a))Q_1(x)$ has a as a root. $P_1(x)$ is a polynomial. Also, $Q'(a) = Q_1(a)$, and

$$P(x)/Q(x) = \frac{P(a)/Q'(a)}{x - a} + \frac{P_1(x)}{Q_1(x)}.$$

Next it must be shown that

$$\frac{P_1(b)}{Q_1'(b)} = \frac{P(b)}{Q'(b)}$$

for every root b of Q_1. Since $Q'(x) = (x-a)Q'_1(x) + Q_1(x)$, $Q'(b) = (b-a)Q'_1(b) + Q_1(b)$, and $P'(b) = (b-a)P'_1(b) + P_1(b)$. Then it follows that $P(b) / Q'(b) = P_1(b) / Q'_1(b)$. Repeating the initial process,

$$P_1(x)/Q_1(x) = \frac{P_1(b)/Q_1'(b)}{x - b} + \frac{P_2(x)}{Q_2(x)}$$

with $$\frac{P_2(c)}{Q_2'(c)} = \frac{P(c)}{Q'(c)}$$
for every root c of $Q_2(x)$. Since the degrees of the polynomials $P(x)$, $P_1(x)$, $P_2(x)$, … , $P_n(x)$ strictly decrease, the formula becomes $$P(x)/Q(x) = \sum_{a \mid Q(a)=0} \frac{P(a)/Q'(a)}{x-a}.$$
By integrating each term, formula (2) is obtained.

6. EXPANDING THE FORMULA [2]

If the roots of the denominator cannot be expressed in a closed form, then the integral cannot be expressed in a closed form. By definition, a field F is a radical extension of $\mathcal{Q}$ if there is a chain of fields $\mathcal{Q} = F_0 \subseteq F_1 \cdots \subseteq F_n = F$ such that for i with $1 \le i \le n$, $F_i = F_{i-1}(u_i)$ with some power of u_i in F_{i-1}. In formula (2), collecting the terms with the same coefficients,
$$\int \frac{P(x)}{Q(x)}\,dx = \sum_i b_i \operatorname{Log} R_i(x) \qquad (4)$$
where $R_i(x)$ is a polynomial. $\int P/Q$ can be expressed in closed form if there is a radical extension F of $\mathcal{Q}$ with b_i in F and $R_i(x)$ in $F[x]$.

Suppose $\int P/Q$ can be expressed in closed form over F. If Q is irreducible over F then $P = CQ'$ for some C in F. The proof follows. Assume P and Q have no common factors and that in formula (4) R_i and R'_i have no common factors. Also, assume that R_i and R_j for $i \neq j$ have no common factors. Then by differentiating formula (4)
$$PR_1 \cdots R_n = Q\sum_i b_i R_1 \cdots R'_i \cdots R_n.$$
Now R_j divides all the summands on the right except $R_1 \ldots R'_j \ldots R_n$, and R_j and $R_1 \ldots R_n$ divide Q. Since P and Q have no common factors, Q divides $R_1 \ldots R_n$ and $Q = C(R_1 \ldots R_n)$ for C in F. This contradicts the assumption that Q is irreducible over F (unless $n = 1$). Hence, $Q = CR_1$ and $P/Q = b_1 R'_1 / R_1$.

7. CONCLUSIONS

The Hermite-Ostrogradski formula is a valuable tool for the integration of polynomials. As has been shown, it offers various benefits, such as computer applications, reduction of computation and integration, and the ability to be expanded. As a result, further research into the use and implementation of the Hermite-Ostrogradski method and its application to the educational environment would be beneficial.

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